Chapter 1

Finite Difference Solution of Maxwell's Equations

1.1 Maxwell's Equations

The principles of electromagnetism have been deduced from experimental observations. These principles are Faraday's law, Ampère's law and Gauss's laws for electric and magnetic fields. James Clerk Maxwell (1831-1879) has discovered the asymmetry in the above laws and introduced a new term in Ampère's law, the so-called *displacement current*. This alteration in Ampère's law provides that a changing electric flux produces a magnetic field, just as Faraday's law provides that a changing magnetic field produces an electric field. The asymmetry in Gauss's laws must be accepted until the existence of magnetic monopole is verified experimentally. The amended relationships of electricity and magnetism are called Maxwell's equations and are usually written in the form

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} - \mathbf{J}_m, \tag{1.1.1}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_e, \tag{1.1.2}$$

$$\nabla(\varepsilon \mathbf{E}) = \varrho, \tag{1.1.3}$$

$$\nabla(\mu \mathbf{H}) = 0, \tag{1.1.4}$$

where

$$\mathbf{E} = (E_x(t,x,y,z), E_y(t,x,y,z), E_z(t,x,y,z)) \tag{1.1.5}$$

is the electric field strength,

$$\mathbf{H} = (H_x(t,x,y,z), H_y(t,x,y,z), H_z(t,x,y,z)) \tag{1.1.6}$$

is the magnetic field strength (t is the time and x, y, z are the spatial coordinates), ε and μ (electric permittivity and magnetic permeability) are material parameters, ϱ is the charge density, and $\mathbf{J}_e$ and $\mathbf{J}_m$ are the electric and magnetic conductive current densities, respectively. The first two equations are called curl equations and the last to are the so-called divergence equations. The solution of Maxwell's equations means the computation of the field strengths using the material parameters and some initial and boundary conditions. For the sake of simplicity we suppose that there are no conductive currents and free charges in the computational domain ($\mathbf{J}_e = \mathbf{0}$, $\mathbf{J}_e = \mathbf{0}$, $\varrho = 0$), thus we have the system

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{1.1.7}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{1.1.8}$$

$$\nabla (\varepsilon \mathbf{E}) = 0, \tag{1.1.9}$$

$$\nabla (\mu \mathbf{H}) = 0, \tag{1.1.10}$$

It is easy to show that if equations (1.1.9) and (1.1.10) are satisfied by the initial conditions, then they are valid automatically at any further time instant. Thus we must solve only the system consisting of equations (1.1.7) and (1.1.8).

1.1.1. Exercise. Show the above statement, that is prove that if equations (1.1.9) and (1.1.10) are valid in the initial state, then these equations can be omitted from the system. □

1.2 Finite Difference Time Domain Method

In real-life problems, the exact solution of system (1.1.1)-(1.1.4) is very complicated or even impossible, this is why numerical methods are generally applied. The first and still extensively applied method is the Finite Difference Time Domain (FDTD) method constructed by K. Yee in 1966. Despite the simplicity of the method, because of its large computational and storage costs, it has been generally applied only since the end of the eighties. In the past fifteen years the number of FDTD publications shows a nearly exponential growth (Figure 1.2.1). J.B. Schneider and K. Shlager maintain a world wide web page (`www.fdtd.org`), where they have catalogued many publications relating to the FDTD method. Nowadays the number of entries is altogether around 4500.

The number of FDTD publications shows that the method has a very wide application area. A very important field is the investigation of the interactions between biological bodies and electromagnetic fields (electromagnetic energy coupled to the human head due to mobile telephones, electromagnetic interference between pacemakers and mobile phones, etc.). Another widely studied area is the modelling of discrete and integrated microwave circuits. Further applications include (but are not limited to): modelling transmission lines; radar cross section prediction; ionospheric and plasma scattering; integrated optics; electromagnetic compatibility; antenna design; etc. These applications can all be supported by the basic FDTD algorithm, requiring only different pre- and post-processing interfaces to obtain the

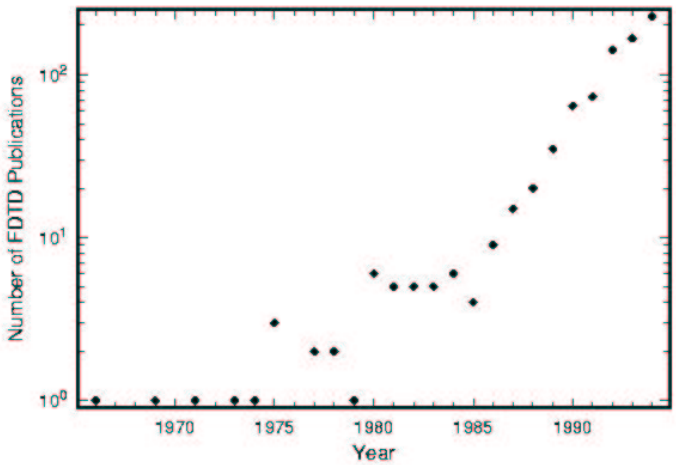

Figure 1.2.1: *Number of FDTD publications from 1966 till 1995. (Source:* J.B. Schneider, K. Shlager*,* A Selective Survey of the Finite-Difference Time-Domain Literature*, IEEE Antennas and Propagation Magazine, vol. 37, no. 4, 1995, pp. 39-56.)*

final results. Figure 1.2.2 shows two special computational results obtained by the FDTD method.

The FDTD method solves the curl equations (1.1.7)-(1.1.8) supposing divergence free initial conditions for the field strengths. The method starts with the definition of a generally rectangular mesh (with the choice of the step-sizes $\Delta x, \Delta y$ and Δz) for the electric field and another staggered (by $\Delta x/2, \Delta y/2$ and $\Delta z/2$) grid for the magnetic field in the computational domain. The building blocks of this mesh are the so-called Yee-cells (see Figure 1.2.3). Let us choose a positive time step Δt, and let us introduce the notation $E_x|_{i,j,k}^n$ for the approximation of the x-component of the electric field at the point $(i\Delta x, j\Delta y, k\Delta z)$ and at the time instant $n\Delta t$. The value of the material parameter ε at the point $(i\Delta x, j\Delta y, k\Delta z)$ is denoted by $\varepsilon_{i,j,k}$ (ε is supposed to be independent of time). The same notations are used for the other field components and material parameters. With the above notations the FDTD method computes the numerical solution as follows.

$$\frac{E_x|_{i+1/2,j,k}^{n+1} - E_x|_{i+1/2,j,k}^{n}}{\Delta t} = \tag{1.2.11}$$

$$= \frac{1}{\varepsilon_{i+1/2,j,k}} \left(\frac{H_z|_{i+1/2,j+1/2,k}^{n+1/2} - H_z|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y|_{i+1/2,j,k+1/2}^{n+1/2} - H_y|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z} \right),$$

$$\frac{E_y|_{i,j+1/2,k}^{n+1} - E_y|_{i,j+1/2,k}^{n}}{\Delta t} = \tag{1.2.12}$$

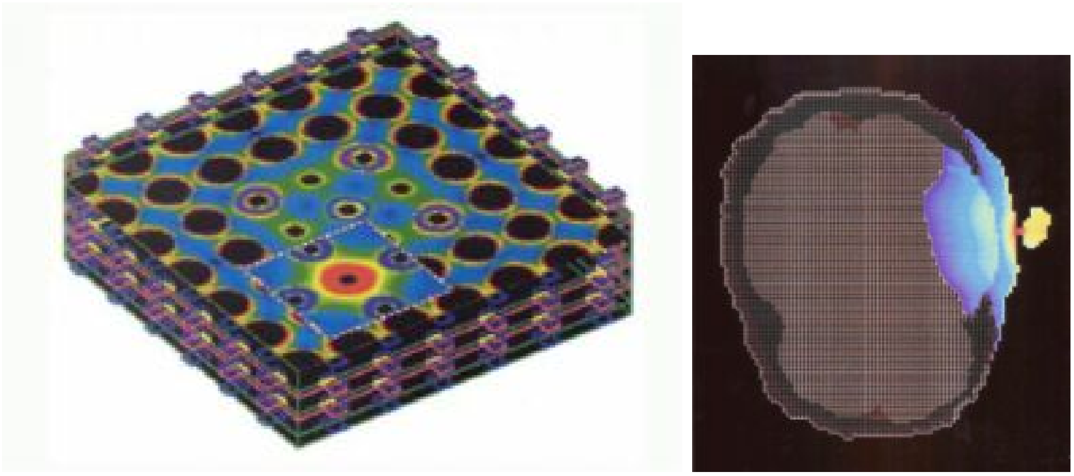

Figure 1.2.2: Left: *Visualization of the FDTD-computed current density for an 8-layer power-distribution structure feeding a high-speed multichip.* Right: *Visualization of the FDTD-computed absorption rate distribution in an ear-level plane for a 1,900-MHz cellular telephone held vertically against a tilted-head model.* (Source: A. Taflove, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, 2 ed., Artech House, Boston, MA, 2000.)

$$= \frac{1}{\varepsilon_{i,j+1/2,k}} \left(\frac{H_x|^{n+1/2}_{i,j+1/2,k+1/2} - H_x|^{n+1/2}_{i,j+1/2,k-1/2}}{\Delta z} - \frac{H_z|^{n+1/2}_{i+1/2,j+1/2,k} - H_z|^{n+1/2}_{i-1/2,j+1/2,k}}{\Delta x} \right),$$

$$\frac{E_z|^{n+1}_{i,j,k+1/2} - E_z|^{n}_{i,j,k+1/2}}{\Delta t} = \tag{1.2.13}$$

$$= \frac{1}{\varepsilon_{i,j,k+1/2}} \left(\frac{H_y|^{n+1/2}_{i+1/2,j,k+1/2} - H_y|^{n+1/2}_{i-1/2,j,k+1/2}}{\Delta x} - \frac{H_x|^{n+1/2}_{i,j+1/2,k+1/2} - H_x|^{n+1/2}_{i,j-1/2,k+1/2}}{\Delta y} \right),$$

$$\frac{H_x|^{n+1/2}_{i,j+1/2,k+1/2} - H_x|^{n-1/2}_{i,j+1/2,k+1/2}}{\Delta t} = \tag{1.2.14}$$

$$= \frac{1}{\mu_{i,j+1/2,k+1/2}} \left(\frac{E_y|^{n}_{i,j+1/2,k+1} - E_y|^{n}_{i,j+1/2,k}}{\Delta z} - \frac{E_z|^{n}_{i,j+1,k+1/2} - E_z|^{n}_{i,j,k+1/2}}{\Delta y} \right),$$

$$\frac{H_z|^{n+1/2}_{i+1/2,j+1/2,k} - H_z|^{n-1/2}_{i+1/2,j+1/2,k}}{\Delta t} = \tag{1.2.15}$$

$$= \frac{1}{\mu_{i+1/2,j+1/2,k}} \left(\frac{E_x|^{n}_{i+1/2,j+1,k} - E_x|^{n}_{i+1/2,j,k}}{\Delta y} - \frac{E_y|^{n}_{i+1,j+1/2,k} - E_y|^{n}_{i,j+1/2,k}}{\Delta x} \right).$$

In the next subsection we investigate the convergence of the method.

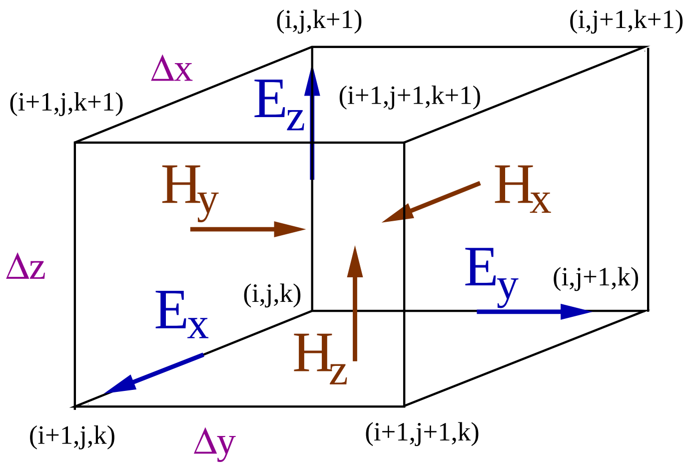

Figure 1.2.3: *The Yee cell.*

Convergence of the FDTD method

The convergence of the method can be guaranteed by the Lax's theorem. According to this theorem we have to show that the method is consistent, then the convergence follows from the stability of the method. With a simple calculation we can obtain that the FDTD method has second order both in time and spatial coordinates.

1.2.1. Exercise. Let us show that the local truncation error of the FDTD method is $O(\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2)$. □

The stability of the method

Divergence-free nature

It is crucial for any numerical solution of the Maxwell's equations to enforce the divergence free property of the fields. We now demonstrate that the FDTD algorithm satisfies (1.1.10) for each cell in the grid, and therefore the grid as a whole. We will approximate the divergence of $\mathbf{E}$ (that is $\nabla\mathbf{E} = \partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z$) in the center of the Yee cell shown in Figure 1.2.3 and at the time instant $(n+1)\Delta t$. (For the sake of simplicity we suppose that ε is independent of the space coordinate).

Thus we have

$$(\nabla\mathbf{E})^{n+1}_{i+1/2,j+1/2,k+1/2} \approx$$

$$\approx \frac{E_x|^{n+1}_{i+1/2,j,k} - E_x|^{n+1}_{i-1/2,j,k}}{\Delta x} + \frac{E_y|^{n+1}_{i,j+1/2,k} - E_y|^{n+1}_{i,j-1/2,k}}{\Delta y} + \frac{E_z|^{n+1}_{i,j,k+1/2} - E_z|^{n+1}_{i,j,k+1/2}}{\Delta z} =$$

$$= \frac{E_x|^{n}_{i+1/2,j,k} - E_x|^{n}_{i-1/2,j,k}}{\Delta x} + \frac{E_y|^{n}_{i,j+1/2,k} - E_y|^{n}_{i,j-1/2,k}}{\Delta y} + \frac{E_z|^{n}_{i,j,k+1/2} - E_z|^{n}_{i,j,k+1/2}}{\Delta z} =$$

$$+\frac{\Delta t}{\varepsilon}\left(\frac{H_z|^{n+1/2}_{i+1/2,j+1/2,k} - H_z|^{n+1/2}_{i+1/2,j-1/2,k}}{\Delta x\Delta y} - \frac{H_y|^{n+1/2}_{i+1/2,j,k+1/2} - H_y|^{n+1/2}_{i+1/2,j,k-1/2}}{\Delta x\Delta z}\right) +$$

$$+\frac{\Delta t}{\varepsilon}\left(\frac{H_y|_{i+1/2,j,k+1/2}^{n+1/2}-H_y|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta x\Delta z}-\frac{H_z|_{i+1/2,j+1/2,k}^{n+1/2}-H_z|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta x\Delta y}\right)+$$

$$+\frac{\Delta t}{\varepsilon}\left(\frac{H_x|_{i,j+1/2,k+1/2}^{n+1/2}-H_x|_{i,j+1/2,k-1/2}^{n+1/2}}{\Delta y\Delta z}-\frac{H_z|_{i+1/2,j+1/2,k}^{n+1/2}-H_z|_{i-1/2,j+1/2,k}^{n+1/2}}{\Delta x\Delta y}\right)+$$

$$+\frac{\Delta t}{\varepsilon}\left(\frac{H_z|_{i+1/2,j-1/2,k}^{n+1/2}-H_z|_{i-1/2,j-1/2,k}^{n+1/2}}{\Delta x\Delta y}-\frac{H_x|_{i,j-1/2,k+1/2}^{n+1/2}-H_x|_{i,j-1/2,k-1/2}^{n+1/2}}{\Delta y\Delta z}\right)+$$

$$+\frac{\Delta t}{\varepsilon}\left(\frac{H_y|_{i+1/2,j,k+1/2}^{n+1/2}-H_y|_{i-1/2,j,k+1/2}^{n+1/2}}{\Delta x\Delta z}-\frac{H_x|_{i,j+1/2,k+1/2}^{n+1/2}-H_x|_{i,j-1/2,k+1/2}^{n+1/2}}{\Delta y\Delta z}\right)+$$

$$+\frac{\Delta t}{\varepsilon}\left(\frac{H_x|_{i,j+1/2,k-1/2}^{n+1/2}-H_x|_{i,j-1/2,k-1/2}^{n+1/2}}{\Delta y\Delta z}-\frac{H_y|_{i+1/2,j,k-1/2}^{n+1/2}-H_y|_{i-1/2,j,k-1/2}^{n+1/2}}{\Delta x\Delta z}\right)=$$

$$=\frac{E_x|_{i+1/2,j,k}^{n}-E_x|_{i-1/2,j,k}^{n}}{\Delta x}+\frac{E_y|_{i,j+1/2,k}^{n}-E_y|_{i,j-1/2,k}^{n}}{\Delta y}+\frac{E_z|_{i,j,k+1/2}^{n}-E_z|_{i,j,k+1/2}^{n}}{\Delta z}.$$

Here we applied the equations of the FDTD iteration, and noticed that all terms disappear except those that approximates the divergence at the time level $n\Delta t$. This shows that the divergence in the center of a Yee cell does not change with the time level. Thus, if the numerical approximation of the divergence of the electric field is zero in the initial state, then the FDTD method produces a divergence-free electric field.

1.2.2. Exercise. Let us show that (1.1.9) is true for the FDTD algorithm. □

Benefits and drawbacks

Now we list the main benefits and drawbacks of the FDTD method, which can be deduced from equations (1.2.11)-(1.2.15) and our above observations. Let us start with the benefits.

- The FDTD method is explicit. (Compare with the Leapfrog method for the one-way wave equation.) We do not have to solve systems of linear equations. The method computes the approximations at the next time-level relatively fast. Easy to implement.
- We have to store only the topical field components on the computer.
- The material parameters can have different values for different field orientations and for different places.
- Divergence conditions fulfil automatically from the discretization.

The drawbacks of the FDTD method are as follows.

Defining the approximations of the field strengths at the points shown in Figure 1.2.3, we calculate the first spatial derivatives in the curl operator using central differences. These approximations of the spatial derivatives produce second order accuracy, this is why this discretization is so common. The methods investigated in this paper all use this type of approximation. The only difference between the methods will be only in the time discretization. In the following we formulate the semi-discretized system.

Let us suppose that the computational space has been divided into N Yee-cells and let us introduce the notation

$$\mathcal{I} = \{(i/2, j/2, k/2) \mid i, j, k \in \mathbb{Z},\ \text{not all odd and not all even},\qquad (1.2.16)$$
$$(i\Delta x/2, j\Delta y/2, k\Delta z/2)^\top \text{ is in the computational domain}\}.$$

We define the functions $\Psi_{i/2,j/2,k/2} : \mathbb{R} \to \mathbb{R}$ $((i/2, j/2, k/2) \in \mathcal{I})$ as

$$\Psi_{i/2,j/2,k/2}(t) = \begin{cases} \sqrt{\varepsilon_{i/2,j/2,k/2}}E_x(t, i\Delta x/2, j\Delta y/2, k\Delta z/2), & \text{if } i \text{ is odd and } j,k \text{ are even},\\ \sqrt{\varepsilon_{i/2,j/2,k/2}}E_y(t, i\Delta x/2, j\Delta y/2, k\Delta z/2), & \text{if } j \text{ is odd and } i,k \text{ are even},\\ \sqrt{\varepsilon_{i/2,j/2,k/2}}E_z(t, i\Delta x/2, j\Delta y/2, k\Delta z/2), & \text{if } k \text{ is odd and } i,j \text{ are even},\\ \sqrt{\mu_{i/2,j/2,k/2}}H_x(t, i\Delta x/2, j\Delta y/2, k\Delta z/2), & \text{if } j,k \text{ are odd and } i \text{ is even},\\ \sqrt{\mu_{i/2,j/2,k/2}}H_y(t, i\Delta x/2, j\Delta y/2, k\Delta z/2), & \text{if } i,k \text{ are odd and } j \text{ is even},\\ \sqrt{\mu_{i/2,j/2,k/2}}H_z(t, i\Delta x/2, j\Delta y/2, k\Delta z/2), & \text{if } i,j \text{ are odd and } k \text{ is even}, \end{cases} \qquad (1.2.17)$$

where $\varepsilon_{i/2,j/2,k/2}$ and $\mu_{i/2,j/2,k/2}$ denote the electric permittivity and magnetic permeability at the points $(i\Delta x/2, j\Delta y/2, k\Delta z/2)^\top$, respectively. This setting corresponds to the staggered grid spatial discretization. Starting from the rearranged form of the Maxwell equations

$$\partial_t(\sqrt{\varepsilon}\mathbf{E}) = \frac{1}{\sqrt{\varepsilon}}\nabla \times \frac{(\sqrt{\mu}\mathbf{H})}{\sqrt{\mu}}, \qquad (1.2.18)$$

$$\partial_t(\sqrt{\mu}\mathbf{H}) = -\frac{1}{\sqrt{\mu}}\nabla \times \frac{(\sqrt{\varepsilon}\mathbf{E})}{\sqrt{\varepsilon}}, \qquad (1.2.19)$$

we can obtain the semi-discretized system

$$\frac{\mathrm{d}\Psi_{i/2,j/2,k/2}(t)}{\mathrm{d}t} = \frac{1}{\varepsilon_{i/2,j/2,k/2}}\left[\frac{\Psi_{i/2,(j+1)/2,k/2}(t)}{\Delta y\sqrt{\mu_{i/2,(j+1)/2,k/2}}} - \frac{\Psi_{i/2,(j-1)/2,k/2}(t)}{\Delta y\sqrt{\mu_{i/2,(j-1)/2,k/2}}} - \frac{\Psi_{i/2,j/2,(k+1)/2}(t)}{\Delta z\sqrt{\mu_{i/2,j/2,(k+1)/2}}} + \frac{\Psi_{i/2,j/2,(k-1)/2}(t)}{\Delta z\sqrt{\mu_{i/2,j/2,(k-1)/2}}}\right], \text{ if } i \text{ is odd and } j,k \text{ are even}, \qquad (1.2.20)$$

$$\frac{\mathrm{d}\Psi_{i/2,j/2,k/2}(t)}{\mathrm{d}t} = \frac{1}{\varepsilon_{i/2,j/2,k/2}}\left[\frac{\Psi_{i/2,j/2,(k+1)/2}(t)}{\Delta z\sqrt{\mu_{i/2,j/2,(k+1)/2}}} - \frac{\Psi_{i/2,j/2,(k-1)/2}(t)}{\Delta z\sqrt{\mu_{i/2,j/2,(k-1)/2}}} - \frac{\Psi_{(i+1)/2,j/2,k/2}(t)}{\Delta x\sqrt{\mu_{(i+1)/2,j/2,k/2}}} + \frac{\Psi_{(i-1)/2,j/2,k/2}(t)}{\Delta x\sqrt{\mu_{(i-1)/2,j/2,k/2}}}\right], \text{ if } j \text{ is odd and } i,k \text{ are even}, \qquad (1.2.21)$$

$$\frac{\mathrm{d}\Psi_{i/2,j/2,k/2}(t)}{\mathrm{d}t} = \frac{1}{\varepsilon_{i/2,j/2,k/2}}\left[\frac{\Psi_{(i+1)/2,j/2,k/2}(t)}{\Delta x\sqrt{\mu_{(i+1)/2,j/2,k/2}}} - \frac{\Psi_{(i-1)/2,j/2,k/2}(t)}{\Delta x\sqrt{\mu_{(i-1)/2,j/2,k/2}}} - \frac{\Psi_{i/2,(j+1)/2,k/2}(t)}{\Delta y\sqrt{\mu_{i/2,(j+1)/2,k/2}}} + \frac{\Psi_{i/2,(j-1)/2,k/2}(t)}{\Delta y\sqrt{\mu_{i/2,(j-1)/2,k/2}}}\right], \text{ if } k \text{ is odd and } i,j \text{ are even,} \tag{1.2.22}$$

$$\frac{\mathrm{d}\Psi_{i/2,j/2,k/2}(t)}{\mathrm{d}t} = \frac{1}{\mu_{i/2,j/2,k/2}}\left[\frac{\Psi_{i/2,j/2,(k+1)/2}(t)}{\Delta z\sqrt{\varepsilon_{i/2,j/2,(k+1)/2}}} - \frac{\Psi_{i/2,j/2,(k-1)/2}(t)}{\Delta z\sqrt{\varepsilon_{i/2,j/2,(k-1)/2}}} - \frac{\Psi_{i/2,(j+1)/2,k/2}(t)}{\Delta y\sqrt{\varepsilon_{i/2,(j+1)/2,k/2}}} + \frac{\Psi_{i/2,(j-1)/2,k/2}(t)}{\Delta y\sqrt{\mu_{i/2,(j-1)/2,k/2}}}\right], \text{ if } j,k \text{ are odd and } i \text{ is even,} \tag{1.2.23}$$

$$\frac{\mathrm{d}\Psi_{i/2,j/2,k/2}(t)}{\mathrm{d}t} = \frac{1}{\mu_{i/2,j/2,k/2}}\left[\frac{\Psi_{(i+1)/2,j/2,k/2}(t)}{\Delta x\sqrt{\varepsilon_{(i+1)/2,j/2,k/2}}} - \frac{\Psi_{(i-1)/2,j/2,k/2}(t)}{\Delta x\sqrt{\varepsilon_{(i-1)/2,j/2,k/2}}} - \frac{\Psi_{i/2,j/2,(k+1)/2}(t)}{\Delta z\sqrt{\varepsilon_{i/2,j/2,(k+1)/2}}} + \frac{\Psi_{i/2,j/2,(k-1)/2}(t)}{\Delta z\sqrt{\mu_{i/2,j/2,(k-1)/2}}}\right], \text{ if } i,k \text{ are odd and } j \text{ is even,} \tag{1.2.24}$$

$$\frac{\mathrm{d}\Psi_{i/2,j/2,k/2}(t)}{\mathrm{d}t} = \frac{1}{\mu_{i/2,j/2,k/2}}\left[\frac{\Psi_{i/2,(j+1)/2,k/2}(t)}{\Delta y\sqrt{\varepsilon_{i/2,(j+1)/2,k/2}}} - \frac{\Psi_{i/2,(j-1)/2,k/2}(t)}{\Delta y\sqrt{\varepsilon_{i/2,(j-1)/2,k/2}}} - \frac{\Psi_{(i+1)/2,j/2,k/2}(t)}{\Delta x\sqrt{\varepsilon_{(i+1)/2,j/2,k/2}}} + \frac{\Psi_{(i-1)/2,j/2,k/2}(t)}{\Delta x\sqrt{\mu_{(i-1)/2,j/2,k/2}}}\right], \text{ if } i,j \text{ are odd and } k \text{ is even,} \tag{1.2.25}$$

which can be written in a shorter form as

$$\frac{\mathrm{d}\Psi(t)}{\mathrm{d}t} = \mathbf{A}\Psi(t),\ t > 0. \tag{1.2.26}$$

The vector-scalar function $\Psi : \mathbb{R} \to \mathbb{R}^{6N}$, $\Psi(t) = (\ldots, \Psi_{i/2,j/2,k/2}(t), \ldots)^\top$ can be obtained from an arbitrary ordering of the functions $\Psi_{i/2,j/2,k/2}$ into a vector and $\mathbf{A} \in \mathbb{R}^{6N\times 6N}$. From equations (1.2.20)-(1.2.25) follow some important properties of $\mathbf{A}$ directly.

Lemma 1.2.3 *Every row of* $\mathbf{A}$ *consists at most four nonzero elements in the forms* $1/(\sqrt{\varepsilon_{.,.,.}\mu_{.,.,.}}\Delta.)$*, that is* $\mathbf{A}$ *is a sparse matrix.* $\mathbf{A}$ *is a skew-symmetric matrix* ($\mathbf{A}^\top = -\mathbf{A}$)*.*

The system (1.2.26) must be solved applying a divergence-free initial condition for $\Psi(0)$. The solution can be written in the form

$$\Psi(t) = \exp(t\mathbf{A})\Psi(0), \tag{1.2.27}$$

where $\exp(t\mathbf{A})$ denotes the exponential matrix and it is well-defined with the Taylor-series of the exponential function. This matrix exponential cannot be computed directly because $\mathbf{A}$ is a very large matrix. According to this representation, usually, the numerical methods for the Maxwell equations are based on some approximation of the matrix exponential $\exp(t\mathbf{A})$. With the choice of a time-step $\Delta t > 0$

$$\Psi(t + \Delta t) = \exp(\Delta t\mathbf{A})\Psi(t) \tag{1.2.28}$$

follows from (1.2.27). Using this equality the one-step iteration

$$\Psi^{n+1} = U_n(\Delta t\mathbf{A})\Psi^n,\ \Psi^0 \text{ is given} \tag{1.2.29}$$

can be defined, where $U_n(\Delta t\mathbf{A})$ is the approximation of the exponential $\exp(\Delta t\mathbf{A})$ (this approximation may depend on n) and Ψ^n is the approximation of the function Ψ at time-level $n\Delta t$.

Contents