Splitting method

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1 Splitting method

The splitting method is one of the most powerful method to solve the abstract Cauchy problems. The main idea is to lead the complex problem to the sequence of sub-problems with simpler structure.

1.1 Introduction of splitting method for ODE' system

In order to give an introduction to the splitting theory, first we consider the Cauchy problem for the system of ordinary differential equations with constant coefficients, i.e.

$$\begin{aligned} \frac{dw}{dt} &= (A+B)w(t), \qquad 0 < t \leq T < +\infty; \\ w(0) &= w_0, \end{aligned} \tag{1.1}$$

where $A, B \in \mathbb{R}^{sxs}$ are given matrices, $w_0 \in \mathbb{R}^s$ is a given vector and the unknown function is $w : [0,T) \to R^s$. Further, we define a usually small number $\tau \ll T$ such that $T = N \cdot \tau$. Then we replace the Cauchy problem

(1.1) on the above defined N sub-intervals with the following sequence of the modified Cauchy problems:

$$\begin{aligned} &\frac{dw_1^n}{dt}(t) = Aw_1^n(t), \qquad (n-1)\tau < t \le n\tau, \\ &w_1^n((n-1)\tau) = w_2^{n\,i\,1}((n-1)\tau), \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &\frac{dw_2^n}{dt}(t) = Bw_2^n(t), \qquad (n-1)\tau < t \le n\tau, \\ &w_2^n((n-1)\tau) = w_1^n(n\tau), \end{aligned} \tag{1.3}$$

$n = 1, 2, \dots N$.

Here $w_2^0(0) = w_0$, and the function $w_{sp}^N(n\tau) = w_2^n(n\tau)$, defined at the points $t_n = n\tau \ \ (n = 0, 1 \dots N)$ is called splitting solution of the problem.

As one can see, the above *splitting process* means the following: we divide the whole time interval into N (equal) parts of the length τ. On each of them we solve two simpler problems, where the different problems are connected via the initial conditions.

Let us consider the first step. The splitting is

$$\begin{aligned} &\frac{dw_1^1}{dt}(t) = Aw_1^1(t), \qquad 0 < t \le \tau \\ &w_1^1(0) = w_0 \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &\frac{dw_2^1}{dt}(t) = Bw_2^1(t), \qquad 0 < t \le \tau \\ &w_2^1(0) = w_1^1. \end{aligned} \tag{1.5}$$

Since the solutions of the Cauchy problems (1.14) and (1.15) are the functions $w_1^1(t) = \exp(At)w_0$ and $w_2^1(t) = \exp(Bt)w_1^1(\tau)$ respectively, therefore,

$$w_{sp}^N(\tau) = \exp(B\tau)\exp(A\tau)w_0. \tag{1.6}$$

At the same time, the solution of the original problem (1.1) at $t = \tau$ is

$$w(\tau) = \exp((A+B)\tau) \cdot w_0. \tag{1.7}$$

We can see that usually $w(\tau) \neq w_{sp}^N(\tau)$, i.e. there is some error, called *local splitting error*, defined as

$$Err_{sp}(\tau) := w(\tau) - w_{sp}^N(\tau) \equiv [\exp(A+B)\tau - \exp(B\tau)\cdot\exp(A\tau)]w_0. \quad (1.8)$$

This shows that the local splitting error vanishes if and only if the condition

$$\exp(A+B)\tau = \exp(B\tau)\cdot\exp(A\tau) \quad (1.9)$$

holds.

Let us compare the two sides of (1.9). According to the definition of the matrix exponential we have

$$\begin{aligned}\exp(A+B)\tau &= I + (A+B)\tau + \frac{1}{2!}(A+B)^2\tau^2 + O(\tau^3) = \\ &= I + (A+B)\tau + \frac{1}{2!}(A^2+B^2)\tau^2 + \\ &+ \frac{1}{2!}(A\cdot B)\tau^2 + \frac{1}{2!}(B\cdot A)\tau^2 + O(\tau^3).\end{aligned} \quad (1.10)$$

On the other hand, we have

$$\begin{aligned}&\exp(B\tau)\cdot\exp(B\tau) = \\ &= (I + B\tau + \frac{1}{2!}B^2\tau^2 + O(\tau^3))\cdot(I + A\tau + \frac{1}{2!}A^2\tau^2 + O(\tau^3)) = \\ &= I + (A+B)\tau + \frac{1}{2!}(A^2+B^2)\tau^2 + BA\tau^2 + O(\tau^3).\end{aligned} \quad (1.11)$$

Comparing the expressions (1.10) and (1.11), the necessary condition of their equality is the commutativity of the matrices A and B, i.e.

$$AB = BA. \quad (1.12)$$

Clearly, under this condition the identity

$$\exp(A+B) = \exp(B)\exp(A) = \exp(A)\exp(B) \quad (1.13)$$

also holds, therefore, we can state the following

Lemma 1.1 *For the splitting (1.2)–(1.3), called sequential splitting, the local splitting error disappears if and only if the condition (1.12) holds, i.e. the operators A and B are commuting.*

We can define other type of the splittings, too. The most popular and widely used is the so called Strang-splitting (or Strang-Marchuk splitting), defined as follows:

$$\begin{aligned} &\frac{dw_1^n}{dt}(t) = Aw_1^n(t), \qquad (n-1)\tau < t \le (n-0.5)\tau, \\ &w_1^n((n-1)\tau) = w_3^{n\,i\,1}((n-1)\tau), \end{aligned} \tag{1.14}$$

$$\begin{aligned} &\frac{dw_2^n}{dt}(t) = Bw_2^n(t), \qquad (n-1)\tau < t \le n\tau, \\ &w_2^n((n-1)\tau) = w_1^n((n-0.5)\tau), \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} &\frac{dw_3^n}{dt}(t) = Aw_3^n(t), \qquad (n-0.5)\tau < t \le n\tau, \\ &w_3^n((n-0.5)\tau) = w_2^n(n\tau), \end{aligned} \tag{1.16}$$

for $n = 1, 2, \ldots N$.
Here $w_3^0(0) = w_0$, and the function $w_{sp}^N(n\tau) = w_3^n(n\tau)$, defined at the points $t_n = n\tau$ $(n = 0, 1 \ldots N)$ is the corresponding splitting solution of the problem.
As an easy computation shows that for the local error of the Strang splitting we get

$$\begin{aligned} &Err_{sp}(\tau) := w(\tau) - w_{sp}^N(\tau) = \\ &= \left[\exp(\tau(A+B)) - \exp(\frac{\tau}{2}A)\cdot\exp(\tau B)\cdot\exp(\frac{\tau}{2}A)\right] w_0. \end{aligned} \tag{1.17}$$

Clearly, if the matrices A and B commute then the splitting error vanishes, i.e. the condition (1.12) is sufficient condition of the relation $Err_{sp}(\tau) = 0$.

Exercise 1.2 *Prove that the condition (1.12) is not the nececessary condition of the relation* $Err_{sp}(\tau) = 0$.

Another alternative is *the symetrically weighted sequantial splitting* (SWSS) which means the following. We execute two sequatial splittings in different

ordering (which are independent tasks) and after each step the splitting approximation is computed as the arithmetical mean of the results. I.e. the algorithm of the computation is the following

$$\begin{gathered}\frac{du_1^n(t)}{dt} = Au_1^n(t), \qquad (n-1) < t \le n\tau, \\ u_1^n((n-1)\tau) = u_2^{n\,i\,1}((n-1)\tau);\end{gathered} \tag{1.18}$$

$$\begin{gathered}\frac{du_2^n(t)}{dt} = Bu_2^n(t), \qquad (n-1) < t \le n\tau, \\ u_2^n((n-1)\tau) = u_1^n(n\tau);\end{gathered} \tag{1.19}$$

and

$$\begin{gathered}\frac{dv_1^n(t)}{dt} = Bv_1^n(t), \qquad (n-1) < t \le n\tau, \\ v_1^n((n-1)\tau) = v_2^{n\,i\,1}((n-1)\tau);\end{gathered} \tag{1.20}$$

$$\begin{gathered}\frac{dv_2^n(t)}{dt} = Av_2^n(t), \qquad (n-1) < t \le n\tau, \\ v_2^n((n-1)\tau) = v_1^n(n\tau);\end{gathered} \tag{1.21}$$

and

$$w_{sp}^N(n\tau) := \frac{u_2^n(n\tau) + v_2^n(n\tau)}{2} \tag{1.22}$$

for $n = 1, 2, \ldots N$, where $u_2^0(0) = v_2^0(0) = w_0$.

One can easily see that under the condition of commutativity of the matrices A and B, i.e. under the condition (1.12), the local splitting error vanishes.

1.2 Splitting as a discretization method

We could see that the use of any splitting to the problem (1.1) in fact results in a discretization process: while the true solution $w(t)$ of (1.1) is defined on the time interval $[0, T]$, the splitting solution $w_{sp}^N(n\tau)$ is defined on the mesh $\Omega_\tau := \{n \cdot \tau,\, n = 0, 1 \ldots N\}$. Clearly, the splitting discretization process can be written in the form

$$\begin{gathered}w^{n+1} = C(\tau)w^n, \\ w^0 = w_0\end{gathered} \tag{1.23}$$

where we have used the notation $w^n = w_{sp}^N(n\tau)$. Here $C(\tau)$ denotes an operator (matrix) which corresponds to the applied discretization. For the different splittings introduced above we can define they directly.

- for the sequential splitting

$$C_{ss}(\tau) = \exp(\tau B)\exp(\tau A); \tag{1.24}$$

- for the Strang splitting

$$C_{st}(\tau) = \exp(\frac{\tau}{2}A)\exp(\tau B)\exp(\frac{\tau}{2}A), \tag{1.25}$$

- and for the SWSS splitting

$$C_{SWSS}(\tau) = \frac{1}{2}\left[\exp(\tau B)\exp(\tau A) + \exp(\tau A)\exp(\tau B)\right]. \tag{1.26}$$

This means that the the local splitting error can be identified with the local (approximation) error. Therefore, it is quite natural to use the following

Definition 1.3 *We say that a splitting is of p-th order accurate if for the local splitting error the relation*

$$\|Err_{sp}(\tau)\| = O(\tau^{p+1}). \tag{1.27}$$

holds.

In the following we define the order of accuracy for the different splittings.

- For the sequential splitting the comparision of (1.10) and (1.11) shows that
$$||Err_{sp}(\tau)|| = O(\tau^2),$$
i.e. the sequential splitting is of first order accurate.

- For the Strang splitting a cumbersome (but simple) calculation shows that
$$||Err_{sp}(\tau)|| = \frac{1}{24}(BA^2+A^2B+4BAB-2ABA-2AB^2-2B^2A)\tau^3+O(\tau^4)$$
for any A and B. Therefore, the Strang splitting has second order accuracy.

Exercise 1.4 *Prove that for the SWSS splitting $p = 2$, i.e. this splitting also has second order accuracy.*

Exercise 1.5 *Let us define the modified SWSS when the arithmetical mean (1.22) is replaced by the linear convex combination of the solutions of the sub-problems, i.e.*

$$w_{sp}^{N}(n\tau) := \theta u_2^n(n\tau) + (1-\theta)v_2^n(n\tau) \tag{1.28}$$

with a fixed $\theta \in [0,1]$, called weighted sequantial splitting (WSS). (Clearly, when $\theta = 0.5$ then WSS mens the SWSS and hence has second order accuracy.) Prove that for $\theta \neq 0.5$ the WSS has first order accuracy.

In virtue of the general theory of abstract numerical method, we can introduce the following

Definition 1.6 *A splitting process to the ACP problem (1.1) is called consistent if for its solution $w(t)$ and for all fixed $t \in [0,T)$ the relation*

$$\lim_{\tau \,!\, 0} \|(\frac{C(\tau)-I}{\tau} - (A+B))w(t)\| = 0 \tag{1.29}$$

holds, where the operator (matrix) $C(\tau)$ represents the splitting process.

Example 1.7 For the sequantial splitting, due to (1.24) and (1.11), we have

$$\left[\frac{C(\tau)-I}{\tau} - (A+B)\right] w(t) = \frac{1}{2}(A^2+B^2+2BA)\tau w(t), \tag{1.30}$$

therefore, the sequantial splitting is consistent on the class of the solution of ODE's system.

Exercise 1.8 *Prove that the Strang splitting, the SWSS and the WSS are consistent on the class of the solution of ODE's system.*

As it folows from the abstract theory of numerical mehods, the consistency itself doesn't yield the convergence. Our aim is to use the Lax theorem. Therefore we introduce the following

Definition 1.9 *We say that a splitting with the operator $C(\tau)$ is stable if there exists a constant $K > 0$ such that the relation*

$$\|C(\tau)^n\| \leq K \tag{1.31}$$

holds for all $n \in \mathbb{N}$ such that $n\tau \leq T$.

We know the necessary condition (Neumann condition) of the stability is

$$\rho(C(\tau)) \leq 1 + C\tau \tag{1.32}$$

for all $\tau \in (0, \tau_0)$. For the symmetric matrices $C(\tau)$ the Neumann condition is necessary and sufficient condition of the stability. We note that the condition

$$\|C(\tau)\| \leq 1 + C\tau \tag{1.33}$$

is a sufficient condition of the stability. The contractivity of the operators $\exp(\tau A)$ and $\exp(\tau B)$, i.e.

$$||\exp(\tau A)|| \leq 1; \quad ||\exp(\tau B)|| \leq 1 \tag{1.34}$$

in some norm, is clearly a sufficient condition of the stability (1.31).

Lemma 1.10 *The sequantial splitting is unconditionally stable for the ODE's system (1.1).*

Proof. Since the relation

$$\begin{aligned} &\|C_{ss}(\tau)^n\| \leq \|C_{ss}(\tau)\|^n \leq [\|\exp(\tau B)\|]^n \, [\|\exp(\tau A)\|]^n \leq \\ &\leq [\exp(\tau\|B\|)]^n \, [\exp(\tau\|A\|)]^n \exp(n\tau\|B\|) \exp(n\tau\|A\|) = \\ &= \exp\left[n\tau(\|A\| + \|B\|)\right] \leq \exp\left[T(\|A\| + \|B\|)\right] \end{aligned} \tag{1.35}$$

hold for all $n \in \mathbb{N}$ such that $n\tau \leq T$, therefore the splitting is stable with the constant $K = \exp\left[T(\|A\| + \|B\|)\right]$. ■

Consequently, the sequantial splitting is unconditionally convergent on the class of the solutions of the ODE's system.

Exercise 1.11 *Analyse the stability and the convergence of the Strang splitting, the SWSS and SWS on the class of the solution of the ODE's system (1.1).*